Rethinking macroeconomic policy within a simple dynamic model

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Abstract

Standard macroeconomics studies phenomena produced in different time horizons (short-run policy, business cycles, or long-run growth) through disjoint models and separate thematic areas. As Solow (2008) or Blanchard (2009) point out, this situation is not satisfactory. We need simple models which bring together macroeconomic phenomena in a coherent temporal framework. The model we propose incorporates short, mid, and long term mechanisms in a relatively simple framework. A joint study of these mechanisms illuminates latent instabilities which could have affected the generation and development of the "Great Recession". Moreover, the properties of our model lead us to reformulate macroeconomic policy problems which were thought to have been solved before the current crisis.

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JEL-Codes: C02, E10, E69.

1.- Introduction

Following a century of debates, revolutions and counter-revolutions in Macroeconomics, the start of the 21st century saw three apparently clear *patterns of macroeconomic policy* (see Blanchard *et al.*, 2010)[2]:

1) According to mainstream thinking in the discipline, the fundamental tool of macroeconomic policy was monetary policy, which had to be managed in an independent, transparent and rigorous way by the Central Banks. The short term control of interest rates and open market operations had to be focussed on maintaining a stable, predictable and reduced inflation rate (e.g. 2%). These objectives were considered to be almost sufficient conditions for macroeconomic stability.
2) Flexibility policies and eliminating market failures (asymmetric information, imperfect competition, externalities) would allow markets to assign resources efficiently and would make it possible for the economic system of advanced economies (considered to be essentially stable) to tend towards an intertemporal dynamic equilibrium. In this

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[2] We leave aside the domain of international macroeconomics since it is a whole field in itself.

way, the real variables per capita would tend to increase at the rate of technical progress and, once the relevant imperfections and rigidities were corrected, the natural unemployment rate would be reduced.

3) Finally, fiscal policy was considered to be inefficient in the long term and distorting in the short term. For this reason, using fiscal policy as a stabilizing policy was not recommended.

This (simplified) vision of the pre-crisis consensus in macroeconomic policy lay, on the one hand, on a theoretical convergence reflected in the construction and use of domain-specific *dynamic stochastic equilibrium models*[3] (Blanchard, 2009). On the other hand, the policy patterns mentioned – which come from these models – seemed to be backed up by facts. Thus, at the end of the 20th century – a century which had seen two world wars, the Great Depression, several long-lasting recessions, a period of stagflation and the Volcker disinflation - the 1987-2006 period was seen as a huge success in terms of macroeconomic management. These two decades of Great Moderation were characterized by the decrease in volatility in sequences of inflation and GDP, moderate and stable levels of inflation, growth rates lower than expected but maintained over time, and acceptable levels of unemployment in most advanced economies (Bernanke, 2013).

This context, favouring theoretical consensus and supported by supposedly efficient economic policies fell apart with the outbreak of the Great Recession in 2007. The Great Moderation was abruptly and unexpectedly interrupted by the most intense worldwide economic upheaval in the last 60 years (Krugman, 2011; Blanchard *et al.*, 2012). The intense and prolonged Great Recession has logically sparked off a debate in Macroeconomics regarding the capacity of the *dynamic stochastic models* to explain the crisis and to restore growth and employment (Solow 2008; Stiglitz, 2011). Moreover, doubts have swiftly come about regarding the validity of the self-complacent convictions on macroeconomic policy held before the crisis. For example, Romer (2012) or Blanchard and Johnson (2013) warn that the Great Recession forces us to revise practices and mechanisms of macroeconomic

[3]Disjoint *domain-specific* models of this type, which differ on the time horizon of the phenomena involved, are (e.g.) the DSGE models of short-run/medium-run fluctuations in Clarida *et al.*, (2000) or Woodford (2003); or the dynamic stochastic equilibrium models of long-run growth in Aghion and Howitt (1998).

policy which we believed we understood before the crisis, but which it now seems that we did not understand so well. Perhaps, as Blanchard (2009) points out, some of the answers to the new questions we face could be found by studying the – little known – mid-term mechanisms which link the short-term with the long-term in macroeconomic models (Fatas-Villafranca *et al.*, 2012).

Bearing this in mind, and looking at the questions considered to be solved in the pre-crisis consensus, but which are in fact presently under debate once again (see Blanchard *et al.*, 2010), we propose to analyze the following subjects in our work:

1) We probably do not completely understand the relations between the inflation rate and actual economic activity. Is the objective of *stable and reduced inflation* a sufficient condition or, perhaps, a necessary but not sufficient condition for macroeconomic stability and growth?
2) What are the relations between the *specific objective of inflation* set by a Central Bank and the intertemporal equilibrium of the real variables of the economic system?
3) There is much we thought we knew, but in fact do not, regarding the *role of fiscal policy*. Is fiscal policy inefficient in the long term and essentially distorting in the short-term?
4) Finally, looking at the intensity and slow return to the tendency seen after the Great Recession, can we state that the Great Moderation was an essentially stable period? Are the possible intertemporal equilibriums underlying the evolution of advanced economies dynamically stable situations? If not, what should we think about the *stability and instability* of our economic systems in the light of recurring major recessions and persistent fluctuations observed in economic series?

To analyze these questions, we propose a demand-driven growth model in discrete time with unemployment, inflation, public expenditure, and endogenous monetary policy. This model dynamically links short, mid, and long term mechanisms. To be more precise, setting out from well-known aggregate functions, the model contains the following theoretical elements (Romer, 2000, 2012; Blanchard, 2005): an equilibrium condition in the goods market; an equation of capital accumulation from endogenous private investment; a Cobb-Douglas production function in which we assume full use of production capacity and adjustment of the employment rate to demand; a dynamic Phillips curve compatible with the hypothesis of a natural rate of unemployment; and a Taylor rule regulating the modulation of the nominal interest rate by the Central Bank. This means, therefore, an *aggregate macrodynamic model*, without explicit microfoundations, synthesizing key interactions between the main variables in a closed economy. The model ends up becoming a system of five difference equations which can be dealt with analytically. Aware that our methodological option is not the most frequent one in standard Macroeconomics[4], we shall devote a few lines to justify our choice.

As Setterfield and Suresh (2013) argue, aggregate models are a legitimate analysis tool - not just in Macroeconomics but also, in general, in social sciences. This opinion is even shared by some of the founding fathers of the microfoundations programme in Macroeconomics: thus, Frydman and Phelps (2013) have recently warned of the risks we run if we adopt an excessively dogmatic position regarding the need for "sound" microfoundations. Peter Howitt (2006) or J.P. Benassy (2011) also justify a certain methodological eclecticism – even while admitting their preference for microfounded models – allowing for simple aggregate models if they are potentially insightful. In fact there have always been authors in favour of using simple, often "ad-hoc", transparent models in Macroeconomics (Krugman, 2000). And now, after the Great Recession, new voices have emerged which also advocate the setting aside of more complex models and the consideration of simple, analytically tractable models which avoid some of the formal excesses of the dynamic stochastic equilibrium models (Solow, 2008; Blanchard, 2009).

In fact, we also favour adjusting the analysis methodology in accordance with the characteristics of the phenomenon under study. Hence, in previous works, we have worked with different microfoundations: we worked with Neoclassical microfoundations in Almudi and Sanchez-Choliz (2011), and in Sanchez-Choliz *et al.* (2008); we used Evolutionary Neo-Schumperian microfoundations in Fatas-Villafranca *et al.* (2009, 2014). In our current work, given the inherent complexity when trying to link short, mid, and long term mechanisms, we prefer to deal with these questions within a simple aggregate macrodynamic model.

[4] Remembering the previously-mentioned regarding *dynamic stochastic equilibrium* models.

Regarding our results, we can state that the model we propose sheds light on questions of Macroeconomic policy which had been (prematurely) considered to be resolved before the Great Recession. The formal analysis of the model shows there are multiple equilibriums with different stability/instability properties. Far from the habitual belief of mainstream models that steady states are "acceptably stable", our formal analysis reveals plentiful sources of instability. It is interesting to point out that, although under certain conditions monetary policy helps to stabilize the economy in our model, we also show that it can become a source of instability in other circumstances. Moreover, in our model it is perfectly feasible that, dependent on the objectives of the Central Bank, the economy can become stuck in situation similar to a liquidity trap, and then the need to apply non-conventional monetary policies will emerge. In this sense, we can state that Central Banks setting specific, stable, very low inflation objectives does not necessarily guarantee a good functioning of economic systems.

Another interesting result is that fiscal policy plays a relevant role in our model, not just in the short term but also when attempting to bring the economy close to one of the multiple (alternative) steady states. Therefore, it is an instrument which can be essential in the long term (contrary to that predicted by mainstream models). We show that very low levels of public expenditure can exaggerate certain instabilities of the system. To be more precise, fiscal policy interacts in our model with factors normally considered to be long term factors (technical progress, population growth, technical production aspects) by determining the parametric configurations for stability/instability of the equilibriums and the specific pattern of dynamic behaviour of the system (dampened oscillations, stability or instability without oscillations , saddle-path type instability, etc.).

Finally, the formal analysis of the stability of the model and development of simulations shows clearly that there are multiple alternative dynamic patterns in our model. There are zones of the parametric space for which the system's equilibriums are unstable (saddle-path type); and others in which one of the equilibriums is asymptotically stable. Therefore, there are parametric configurations for which the system seems to become stable, but suddenly it enters in a regime of oscillating instability and increasing volatility. This is due to the existence of some stable paths in a context of saddle-path type instability. These characteristics of the dynamics lead us to a possible reinterpretation of the period known as the Great Moderation (1987-2006) and a new way to analyze the causes and possible feasible policies in the context of the Great Recession.

In accordance with all the above-mentioned, we shall organize our work in the following way. In Section 2, we shall present our model. In Sections 3 and 4 we will study the existence and stability of intertemporal dynamic equilibriums, which allows us to characterize important aspects of the resultant dynamics of the system. In Section 5, we shall study what we call the indecomposability of the model's dynamics. In Section 6, we strengthen some of the results with simulations. Finally, in Section 7, we extract implications for economic policy and conclude by relating the formal results of our model with some of the open questions we have pointed out. The fact that our model is relatively simple, tractable, and transparent allows us to deduce a large amount of implications from our formal results. As we shall see, the interaction between short, mid, and long term mechanisms is essential in terms of the effectiveness of economic policies and the stability of the system, and help us to revise some of the recent episodes of economic history from a new perspective.

2.-The Model

2.1.- Overview

In this work we propose a model of growth with unemployment, in discrete time, for a closed economy. In this model, firms produce according to what demand dictates and accumulate capital via the endogenous evolution of investment. Production is carried out based on a Cobb-Douglas production function – with its habitual properties – in which firms use their capacity fully and adjust their employment to their needs. Inflation evolves endogenously following a dynamic Phillips curve compatible with the natural rate hypothesis. Production and inflation dynamics condition the endogenous modulation of monetary policy. This, in turn, affects investment by modifying interest rates. We shall assume exogenous technical progress[5] increasing the efficiency of work, and a growth of the active population (and total population) at a constant rate. To be more specific, we suppose that the total and active population in t is:

$$N_t = N_0(1+n)^t, \quad n \in (0,1)$$

And that the technical progress is shown by:

$$A_t = A_0(1+\delta)^t, \quad \delta \in (0,1)$$

From now on we shall use a combined indicator of these growth rates which will be:

$$\alpha = n + \delta + \delta n, \quad \alpha \in (0,1)$$

Furthermore, we shall express the variables in terms per capita and adjusted by the level of technical progress. Small-case letters will indicate that, in each case, the corresponding variable has been divided by $A_t N_t$ and we shall refer to these variables as "normalized variables". Likewise, we shall consider that decisions are taken at the end of each time period and their effect is seen in the following period (*end period* hypothesis; Turnovsky, 1977)[6].

2.2.- Demand

The aggregate demand (y) - in normalized terms - has three components: consumption (c) investment (i) and public spending (g):

$$y_t = c_t + i_t + g, \qquad g > 0 \tag{1}$$

By assuming a *constant* g (that is, constant normalized public spending), we assume that the real public spending is growing at a constant rate "α", which, as we shall see, is the rate at which the income grows in all the dynamic equilibriums of the model. This is a simplifying supposition but it can help to generate realistic situations in which the public spending/GDP ratio tends to be constant[7].

The *consumption function* has the following expression (in normalized units):

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[5]We make this supposition for simplicity. Even so, the model consists of a system of five difference equations. In other works we have endogenized technical change (see e.g. Fatas-Villafranca *et al*., 2009, 2012; Almudi *et al*., 2013).

[6] It is interesting to note that, in our model, the final period hypothesis is equivalent to supposing that agents form *naive* –therefore *endogenous backward-looking*- expectations.

[7]These would be situations comparable to a moderate version of the so-called Wagner's Law (see Bernanke, 2013). Elsewhere (Fatas-Villafranca *et al*., 2011), we have proposed a model in which the desired level of public spending evolves endogenously starting out from the process of public opinion formation.

$$c_t = \frac{b}{1+\alpha} y_{t-1}, \qquad b \in (0,1) \tag{2}$$

where b is the propensity to consume[8].

We shall suppose that *investment* depends on the real interest rate, $r_{t-1} = R_{t-1} - \pi_{t-1}$, and on the income from the previous period. With normalized units this will respond to the following expression:

$$i_t = \frac{1}{1+\alpha}(\beta - \theta r_{t-1}) y_{t-1}, \quad \theta > 0,\ 0 < \beta < 1, \quad \frac{\beta - (1-b)}{\theta} < r_t. \tag{3}$$

Let us suppose that there is no depreciation. The final inequality of (3) guarantees that income in equilibrium cannot be negative. We suppose that the propensity to invest depends in a negative and lineal way on the real interest rate.

Substituting (2) and (3) in (1) we obtain:

$$y_t = \frac{b}{1+\alpha} y_{t-1} + \frac{1}{1+\alpha}(\beta - \theta r_{t-1})\, y_{t-1} + g \tag{4}$$

2.3.- Production

We shall suppose that the producing sector adjusts its production to the level laid down by demand in accordance with (4). Furthermore, we shall suppose that the productive sector produces in accordance with a *Cobb-Douglas production function* with constant returns to scale, which incorporates *labour-saving technical change*, so:

$$Y_t = K_t^{\mu} (A_t L_t)^{1-\mu}, \qquad \mu \in (0,1)$$

K_t is the capital stock and L_t the level of employment at any time. We shall assume full use of the productive capacity and define the employment rate as $\varepsilon_t = \frac{L_t}{N_t}$.

It is possible to express this function in normalized units, leaving us with:

$$y_t = k_t^{\mu} \varepsilon_t^{1-\mu}, \qquad \mu \in (0,1) \tag{5}$$

Note that if y_t is determined by (4), and the accumulation of k_t is determined by investment

[8]The kinds of consumption and investment functions we propose have a long tradition in Macroeconomics (from e.g. Allen, 1967 to Gandolfo, 2009). We are aware of the existence of more complicated functions. In earlier works we have endogenized the propensity to consume and invest in very different theoretical frameworks (see Fatas-Villafranca *et al*., 2007 or Sanchez-Choliz *et al*., 2008). For the reasons explained in the introduction, on this occasion, we choose the simplest spending functions.

(3), then equation (5) allows us to ensure the employment rate is determined endogenously for each moment with $\varepsilon_t = \left(\frac{y_t}{k_t^{\mu}}\right)^{\frac{1}{1-\mu}}$.

2.4.- Capital accumulation

We shall obtain the *dynamics of the capital in normalized units*, $k_t = \frac{K_t}{A_t N_t}$.

We know that, if depreciation is null, the *investment in normalized units* is:

$$i_t = \frac{K_{t+1} - K_t}{A_t N_t}$$

Therefore, it is clear that:

$$k_{t+1} = \frac{i_t + k_t}{1+\alpha} \tag{6}$$

2.5.- The Phillips curve

We propose that the *inflation rate* evolves following a dynamic *Phillips curve* such as (Blanchard, 2005; Blanchard and Johnson, 2013)[9]:

$$\pi_t = -\gamma + \rho\varepsilon_{t-1} + \pi_{t-1} \qquad \rho > \gamma > 0 \tag{7}$$

2.6.- Monetary policy rule

Finally, we can suppose that monetary authority (*Central Bank*) determines the nominal interest rate in accordance with a possible version of *Taylor's rule* such as (Taylor, 1998, 1999):

$$R_t = R_{t-1} + a_\pi\left(\pi_{t-1} - \pi^*\right) + a_y\left(y_{t-1} - y^*\right), \quad 0 < a_y < a_\pi \tag{8}$$

where π^* and y^* are the objectives for inflation and national income at any time.

We are not supposing that π^* or y^* are necessarily the values of a dynamic equilibrium; they are simply reference objectives for monetary policy. What is implicitly assumed is that the possible equilibriums verify:

[9] In Fatas-Villafranca *et al*. (2012; 2014) we analyzed in more detail questions related to the labour market, salary formation and their aggregate effects. Given the complexity of the phenomena studied on this occasion, we have opted to use a plausible formal treatment but as stylized as possible.

$$a_\pi \pi + a_y y = a_\pi \pi^* + a_y y^* \Leftrightarrow \frac{y - y^*}{\pi - \pi^*} = -\frac{a_\pi}{a_y}$$

2.7.- The model equations

The dynamics of the model can be synthesized, from the previous equations, in the following system of difference equations:

$$\left.\begin{aligned}
k_{t+1} &= \frac{1}{1+\alpha}(k_t + i_t) \\
y_{t+1} &= \frac{b}{1+\alpha} y_t + \frac{1}{1+\alpha}[\beta - \theta(R_t - \pi_t)]y_t + g \\
i_{t+1} &= \frac{1}{1+\alpha}[\beta - \theta(R_t - \pi_t)]y_t \\
\pi_{t+1} &= -\gamma + \rho\left(\frac{y_t}{k_t^\mu}\right)^{\frac{1}{1-\mu}} + \pi_t \\
R_{t+1} &= R_t + a_\pi(\pi_t - \pi^*) + a_y(y_t - y^*)
\end{aligned}\right\} \qquad (9)$$

The system consists of a process of capital accumulation starting out from endogenous investment, an equilibrium equation in the goods market, a Phillips curve compatible – as we shall see – with the hypothesis of natural rate and a monetary policy rule. This system allows for the analysis of macroeconomic policy mentioned in our introduction. Results can be obtained regarding the short, mid and long term evolution of output, employment, inflation and interest rates, as well as the effects of fiscal and monetary policy. Likewise it is interesting to study the determinants of greater or lesser stability and instability of the economy. In the following section, we shall analyze the properties of the model.

3.- Steady State I: Existence and Multiplicity

Any *steady state* of the previous dynamic system will be characterized by the constant values $\{k, y, i, \pi, R\}$, solutions of the resulting system of equations when in (9) $k_{t+1} = k_t = k, y_{t+1} = y_t = y, i_{t+1} = i_t = i, \pi_{t+1} = \pi_t = \pi, R_{t+1} = R_t = R,$ that is to say, characterized by the solution values of the system of equations:

$$\left.\begin{aligned}
&k = \frac{1}{1+\alpha}(k+i) \\
&y = \frac{b}{1+\alpha}y + \frac{1}{1+\alpha}[\beta - \theta(R-\pi)]y + g \\
&i = \frac{1}{1+\alpha}[\beta - \theta(R-\pi)]y \\
&\pi = -\gamma + \rho\left(\frac{y}{k^{\mu}}\right)^{\frac{1}{1-\mu}} + \pi \\
&R = R + a_{\pi}(\pi - \pi^{*}) + a_{y}(y - y^{*})
\end{aligned}\right\} \qquad (10)$$

3.1.- Real variables in steady state

From the fourth equation of (10), we can deduce:

$$\pi = -\gamma + \rho\left(\frac{y}{k^{\mu}}\right)^{\frac{1}{1-\mu}} + \pi \Rightarrow \varepsilon = \left(\frac{y}{k^{\mu}}\right)^{\frac{1}{1-\mu}} = \frac{\gamma}{\rho} \Rightarrow y = \left(\frac{\gamma}{\rho}\right)^{1-\mu} k^{\mu} \qquad (11)$$

In steady state, the *employment rate* is the natural rate – for which the inflation rate will remain constant. The natural rate is given by institutional factors underlying parameters ρ and γ (mechanisms of negotiation in the labour market and transmission of prices). Furthermore, as seen in (11), *production* in steady state is determined by the natural rate of employment ε and by the capital stock *k*.

The first equation of (10) allows us to obtain a necessary relationship in the equilibrium between capital and investment:

$$k = \frac{1}{1+\alpha}(k+i) \Rightarrow k\left(1 - \frac{1}{1+\alpha}\right) = \frac{1}{1+\alpha}i \Rightarrow i = k\alpha \qquad (12)$$

This means that *investment* in steady state is the replacement investment; that is, the necessary investment to maintain capital constant - in normalized units -, given the exogenous growth rate α.

Obtaining the *capital* in the steady state is very important because it allows us to obtain the values of production and investment. Furthermore, as we shall see below, it will also give us the value for the other variables in the system. To obtain k in the steady state we start out from the second and third equations of (10) and we consider (11) and (12):

$$\left.\begin{array}{l} y\left(1-\dfrac{b}{1+\alpha}\right)-g=\dfrac{1}{1+\alpha}\left[\beta-\theta\left(R-\pi\right)\right]y \\ k\alpha=\dfrac{1}{1+\alpha}\left[\beta-\theta\left(R-\pi\right)\right]y \end{array}\right\}\Rightarrow k\alpha=\left(\frac{\gamma}{\rho}\right)^{1-\mu}k^{\mu}\left(1-\frac{b}{1+\alpha}\right)-g$$

from where the following equation is obtained, which must be fulfilled by any steady state k:

$$\left(1-\frac{b}{1+\alpha}\right)\left(\frac{\gamma}{\rho}\right)^{1-\mu}k^{\mu}=\alpha k+g \tag{13}$$

This expression is the savings-investment equilibrium condition (and goods market equilibrium condition) in the steady state. Similarities can be seen between this particular steady state condition and the Solow-Swan Neoclassical growth model. We see in (13) how the savings rate multiplied by natural income (k function via Cobb-Douglas evaluated in the natural employment rate, -see (5)), must be equal to the replacement investment plus public spending. If we assume, as a specific case in (13), that there is no public sector and we have full employment, we would have the Solow model steady state. However, as we shall see later, the dynamics of our model are not ruled by the Solow mechanisms, but by the interaction among short, mid, and long term mechanisms.

Returning to our model, we reconsider equation (13). To determine whether (13) has a solution and, in such a case, how many solutions it has (uniqueness or multiplicity of steady states), we shall begin by studying the properties of the function:

$$h(k)=Tk^{\mu},\ \text{with}\quad T=\left(1-\frac{b}{1+\alpha}\right)\left(\frac{\gamma}{\rho}\right)^{1-\mu}.$$

Function $h(k)$ is the level of normalized savings in the natural situation of the economy. As it is fulfilled that

$$h(0)=0;\lim_{k\to+\infty}h(k)=+\infty;h'(k)=\mu Tk^{\mu-1}>0;h''(k)=\mu(\mu-1)Tk^{\mu-2}<0\ ;$$
$$\lim_{k\to0^{+}}h'(k)=+\infty;\ \lim_{k\to+\infty}h'(k)=0;$$

we can confirm that $h(k)$ is concave with an infinite slope at the origin which decreases as values move towards infinity, tending to be horizontal.

Looking at (13) and considering the properties of $h(k)$, we can see that the financing capacity of the private sector (once the replacement investment is covered) is initially growing until it reaches point k^M, and then decreasing. Furthermore, in general, *there are two "k"* of *steady state* in the model (which vary according to the parametric configuration). All this can be seen in Fig. 1.

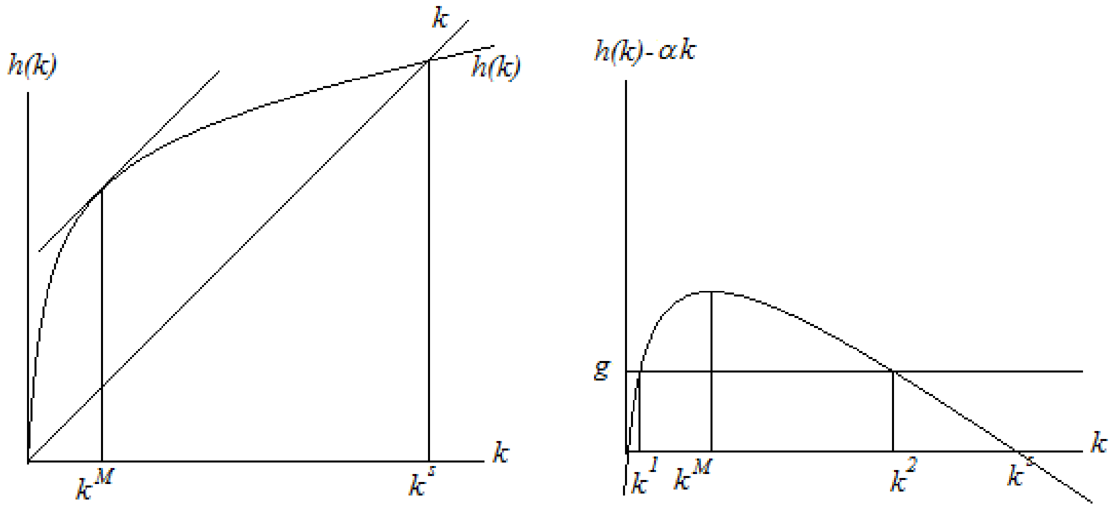

Figure 1.- Existence and Multiplicity of Steady States

To see this in more detail, we shall limit the values of *k* which are possible steady states (bearing in mind (13), *g*>0, and knowing that – in a closed economy - "*g*" cannot be higher than the maximum financing capacity of the private sector). For all this, the possible equilibriums will be located between the two values of *k*, ($k=0$ and $k^S>0$) which verify $Tk^{\mu}-\alpha k=0$. We can affirm that all the possible *k* of equilibrium will be between $k \in \mathrm{D}=(0,k^S)$.

The value k^S is easy to calculate:

$$T.\left(k^s\right)^{\mu}-\alpha k^s=0 \Rightarrow k^s=\left(\frac{T}{\alpha}\right)^{\frac{1}{1-\mu}}=\left[\frac{1}{\alpha}\left(1-\frac{b}{1+\alpha}\right)\right]^{\frac{1}{1-\mu}}\frac{\gamma}{\rho}$$

$$k^s=\left(\frac{\sigma}{\alpha}\right)^{\frac{1}{1-\mu}}.\ \varepsilon$$

where $\sigma = 1 - \frac{b}{1+\alpha}$ is the savings ratio, and ε is the natural employment rate. Note again that, by assuming g=0 to obtain k^s, if we suppose $\varepsilon = 1$, and we ignore the Phillips' curve and Taylor's rule, $k^s = \left(\frac{\sigma}{\alpha}\right)^{\frac{1}{1-\mu}}$ is a Solow-type steady state. However, as we have seen and shall check later on, our model generates steady states with equilibrium properties and dynamic mechanisms very different to the Solow model.

It is also easy to obtain the value k^M, for which the function $z(k) = h(k) - \alpha k = Tk^{\mu} - \alpha k$ reaches its maximum value:

$$z'(k) = \mu T k^{\mu-1} - \alpha = 0 \Rightarrow k^M = \left(\frac{\mu T}{\alpha}\right)^{\frac{1}{1-\mu}} = \left[\frac{\mu}{\alpha}\left(1 - \frac{b}{1+\alpha}\right)\right]^{\frac{1}{1-\mu}} \frac{\gamma}{\rho} = \mu^{\frac{1}{1-\mu}} k^s$$

$$k^M = \mu^{\frac{1}{1-\mu}} k^s$$

with the maximum value of $z(k)$ as:

$$z(k^M) = T.(k^M)^{\mu} - \alpha k^M = T\left(\frac{\mu T}{\alpha}\right)^{\frac{\mu}{1-\mu}} - \alpha\left(\frac{\mu T}{\alpha}\right)^{\frac{1}{1-\mu}} = \left(\frac{T}{\alpha}\left(\frac{\mu T}{\alpha}\right)^{-1} - 1\right)\alpha\left(\frac{\mu T}{\alpha}\right)^{\frac{1}{1-\mu}}$$

$$= \alpha\frac{1-\mu}{\mu} k^M = \alpha\frac{1-\mu}{\mu}\mu^{\frac{1}{1-\mu}} k^S$$

Expression $\alpha\frac{1-\mu}{\mu}k^M$ is the maximum financing capacity of the private sector in steady state (after having carried out its own investment). Thus, looking at (13), it is logical that:

$$g \le z\left(k^M\right) = \alpha\frac{1-\mu}{\mu}k^M \qquad (14)$$

To sum up, returning to Fig.1 and considering (13) we can see two possibilities in terms of *number of steady states in the model*: (1) there will be two possible steady state k, $k^1 < k^M < k^2$, for each value of g which verifies inequality (14) in a strict way; and (2) there will be just one value of steady state k, (k^M), for the specific case of equality in (14). This throws up three comments: (i) it is very relevant that the level of public spending will end up being an essential factor for the existence and stance of the steady states. Thus, fiscal policy conditions the possible long term equilibrium situations – contrary to what is usually stated in standard macroeconomics. This also affects, as we shall see, the short and mid term evolution of the system. (ii) On the other hand, there are (in general) two alternative steady state *k*: in ones where $k < k^M$, (13) is verified for a lower level of saving and for a lower level of investment. In steady states higher than k^M , although the necessary replacement investment is high, the economy is capable of generating more financing. (iii) Finally, for $g>0$, we have two possible kinds of *k*-steady states: those where $k < k^M$ in which the relative weight in the total income of public spending is much higher, given the reduced amount of income and private investment; and those *k* higher than k^M , in which private demand has a relatively higher weight within the total demand. In later sections of the paper, we shall check the relevance of these comments for our stability analysis.

Continuing with the characterization of steady states, with the private sector steady state investment being exactly the replacement investment, and given that we are in natural income, from (10), (11) and (12) we get:

$$k\alpha = \frac{1}{1+\alpha}[\beta - \theta(R-\pi)]\left(\frac{\gamma}{\rho}\right)^{1-\mu} k^{\mu} \Leftrightarrow \frac{k\alpha}{k^{\mu}\left(\frac{\gamma}{\rho}\right)^{1-\mu}} = \frac{\beta - \theta(R-\pi)}{1+\alpha} \Leftrightarrow \frac{k\alpha}{y} = \frac{\beta - \theta(R-\pi)}{1+\alpha}$$

That is, the *investment rate* in steady state (replacement investment/natural income) must coincide with the propensity to invest. This relationship allows us to obtain the steady state value for *real interest rate "r"*. This variable (remember $r = R - \pi$) can be obtained from the previous expressions and is a function of *k* and certain parameters:

$$r = \frac{1}{\theta}\left(\beta - \alpha(1+\alpha)k^{1-\mu}\left(\frac{\gamma}{\rho}\right)^{\mu-1}\right) = \frac{1}{\theta}\left(\beta - \frac{(1+\alpha)\alpha k}{y}\right) = \frac{1}{\theta}\left(\beta - \frac{\alpha(1+\alpha)}{\left(y/k\right)}\right) \qquad (15)$$

The real interest rate in steady state must be lower, the higher the necessary replacement investment, the higher the rate of technical progress and the lower the natural income. In addition, *r* is higher, the higher the average capital productivity in steady state. These are supply factors of the economy; but in (15) it can be seen that r also depends on demand parameters like β and θ.

This finalizes the characterization of real variables in steady states. Note that the features of the steady state reflect both the demand-driven Keynesian content of the model, as well as certain classical/neo-classical characteristics habitually found in literature. Thus, for example, we have seen that public spending plays a (Keynesian) role in determining the equilibrium, but we also find that the steady states are typical neo-classical situations with a natural employment rate, and Solowian features.

3.2.- Nominal variables in steady state

Regarding the nominal variables, taking into account the fifth equation of (10), the following must be verified in the equilibrium:

$$a_\pi\left(\pi-\pi^*\right)+a_y\left(y-y^*\right)=0 \Rightarrow \pi=\pi^*-\frac{a_y}{a_\pi}\left(y-y^*\right)$$

By also considering (11), we can deduce the value of *inflation rate* π in steady state:

$$\pi=\pi^*-\frac{a_y}{a_\pi}\left[\left(\frac{\gamma}{\rho}\right)^{1-\mu}k^\mu-y^*\right] \tag{16}$$

It is very important to note that, according to (16), as the income objective of the Central Bank differs from the natural income of the steady state, the equilibrium inflation will differ from the inflation objective of the Central Bank. Only in the case where monetary policy establishes the steady state value of "*y*" as an income objective (difficult to achieve in imprecise and uncertain environments where the value of "*y*" is not necessarily known in reality), will the equilibrium inflation coincide with the value desired by the Central Bank.

To be specific, eq.(16) shows that it is very likely – in general – that the Central Bank perceives discrepancies between the current inflation rate and its objective. If it were not aware of this fact, even though the economy were in steady state, the monetary authority would have incentives to modify the nominal interest rate to try and achieve its inflation objective; this act would take the economy out of its steady state, and only the stability analysis will tell us whether the economy would tend to return to it.

On the other hand, from (15) and (16) the steady state value of *R* is obtained as the sum of the inflation and the real interest rate of equilibrium:

$$R = \pi^* - \frac{a_y}{a_\pi}\left[\left(\frac{\gamma}{\rho}\right)^{1-\mu} k^\mu - y^*\right] + \frac{1}{\theta}\left[\beta - \alpha(1+\alpha)k^{1-\mu}\left(\frac{\gamma}{\rho}\right)^{\mu-1}\right] \tag{17}$$

Regarding (17), note that if the income objective of the Central Bank were near that of the steady state, and we had a value of *r* near to "0", then establishing an excessively low inflation objective could generate equilibrium values of *R* near to 0. If this situation occurs, conventional monetary policy via the regulation of *R* loses its capacity to act and the economy could reach a steady state dangerously close to a liquidity trap.

3.3.- To sum up

The expression of the equations which define a steady state are:

$$\left.\begin{aligned}
&\left(1-\frac{b}{1+\alpha}\right)\left(\frac{\gamma}{\rho}\right)^{1-\mu} k^\mu = \alpha k + g \\
&y = \left(\frac{\gamma}{\rho}\right)^{1-\mu} k^\mu \\
&i = \alpha k \\
&\pi = \pi^* - \frac{a_y}{a_\pi}\left[\left(\frac{\gamma}{\rho}\right)^{1-\mu} k^\mu - y^*\right] \\
&R = \pi^* - \frac{a_y}{a_\pi}\left[\left(\frac{\gamma}{\rho}\right)^{1-\mu} k^\mu - y^*\right] + \frac{1}{\theta}\left[\beta - \alpha(1+\alpha)k^{1-\mu}\left(\frac{\gamma}{\rho}\right)^{\mu-1}\right]
\end{aligned}\right\} \tag{18}$$

where it must be verified that:

$$\left.\begin{aligned}
&k \in D = \left(0, \left[\frac{1}{\alpha}\left(1-\frac{b}{1+\alpha}\right)\right]^{\frac{1}{1-\mu}} \frac{\gamma}{\rho}\right) \\
&0 < g \le (1-\mu)\left(\frac{\mu}{\alpha}\right)^{\frac{\mu}{1-\mu}} \left(1-\frac{b}{1+\alpha}\right)^{\frac{1}{1-\mu}} \frac{\gamma}{\rho}
\end{aligned}\right\} \tag{19}$$

and the following must also be fulfilled:

$$\left.\begin{aligned} r &= R-\pi = \frac{1}{\theta}\left[\beta - \alpha(1+\alpha)k^{1-\mu}\left(\frac{\gamma}{\rho}\right)^{\mu-1}\right] \\ \varepsilon &= \left(\frac{y}{k^{\mu}}\right)^{\frac{1}{1-\mu}} = \frac{\gamma}{\rho} \end{aligned}\right\} \qquad (20)$$

Note that the steady state situations also verify *three other properties:*

1) Given that normalized variables are constant in steady state, the model has equilibriums in which production, investment, consumption and capital all grow at rate α, and, in terms per capita, they grow at the rate of technical progress.

2) The dominium D of possible steady states k is reduced when the growth rate α grows. Thus:

$$\frac{\partial k^S}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\left[\frac{1}{\alpha}\left(1-\frac{b}{1+\alpha}\right)\right]^{\frac{1}{1-\mu}}\frac{\gamma}{\rho}\right) = \frac{\gamma}{\rho}\frac{1}{1-\mu}\left[\frac{1}{\alpha}\left(1-\frac{b}{1+\alpha}\right)\right]^{\frac{\mu}{1-\mu}}\left[-\frac{1}{\alpha^2}+\frac{b(1+2\alpha)}{\alpha^2(1+\alpha)^2}\right]$$

$$=\frac{\gamma}{\rho}\frac{1}{1-\mu}\left[\frac{1}{\alpha}\left(1-\frac{b}{1+\alpha}\right)\right]^{\frac{\mu}{1-\mu}}\frac{1}{\alpha^2}\left[\frac{-1-2\alpha-\alpha^2+b+2\alpha b}{(1+\alpha)^2}\right]$$

$$=\frac{\gamma}{\rho}\frac{1}{1-\mu}\left[\frac{1}{\alpha}\left(1-\frac{b}{1+\alpha}\right)\right]^{\frac{\mu}{1-\mu}}\frac{1}{\alpha^2}\frac{(-1)\left[(1-b)+\alpha^2+2\alpha(1-b)\right]}{(1+\alpha)^2}<0$$

3) The dominium D of possible k values for a steady state will grow when the natural employment rate grows:

$$\frac{\partial}{\partial \varepsilon}\left[\left[\frac{1}{\alpha}\left(1-\frac{b}{1+\alpha}\right)\right]^{\frac{1}{1-\mu}}\varepsilon\right]>0$$

4.- Steady State II: Stability analysis

The dynamic stability of a steady state (k, y, i, π, R) of (9) is determined by the eigenvalues of the Jacobian matrix of this system, evaluated in the equilibrium.

The Jacobian matrix of (9) is given by:

$$J(k,y,i,\pi,R)=\begin{pmatrix} \frac{1}{1+\alpha} & 0 & \frac{1}{1+\alpha} & 0 & 0 \\ 0 & \frac{b}{1+\alpha}+\frac{\beta-\theta(R-\pi)}{1+\alpha} & 0 & \frac{\theta}{1+\alpha}y & -\frac{\theta}{1+\alpha}y \\ 0 & \frac{\beta-\theta(R-\pi)}{1+\alpha} & 0 & \frac{\theta}{1+\alpha}y & -\frac{\theta}{1+\alpha}y \\ \rho\frac{-\mu}{1-\mu}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}} & \rho\frac{1}{1-\mu}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}} & 0 & 1 & 0 \\ 0 & a_y & 0 & a_\pi & 1 \end{pmatrix}$$

which, in the equilibrium corresponding to a k solution of (13) will be:

$$J=J(k,y(k),i(k),\pi(k),R(k))=$$

$$\begin{pmatrix} \frac{1}{1+\alpha} & 0 & \frac{1}{1+\alpha} & 0 & 0 \\ 0 & \frac{b}{1+\alpha}+\alpha\frac{k}{y} & 0 & \frac{\theta}{1+\alpha}y & -\frac{\theta}{1+\alpha}y \\ 0 & \alpha\frac{k}{y} & 0 & \frac{\theta}{1+\alpha}y & -\frac{\theta}{1+\alpha}y \\ \frac{-\mu\rho}{1-\mu}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}} & \frac{\rho}{1-\mu}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}} & 0 & 1 & 0 \\ 0 & a_y & 0 & a_\pi & 1 \end{pmatrix} \quad (21)$$

where $y=\left(\frac{\gamma}{\rho}\right)^{1-\mu}k^{\mu}$.

4.1.- The instability of the steady states for $k<k^M$

One condition which ensures the (local) stability of the equilibrium point $(k,y(k),i(k),\pi(k),R(k))$ is that all the eigenvalues of the characteristic equation of matrix (21) have a modulus below one. If one of the eigenvalues has modulus above a value of one, this is sufficient cause for the steady state to be unstable. Based on this, we can establish the following proposition:

Proposition 1.- *The following is verified:*

1) *If there are two steady states, the one associated to the lower value of k ($k^1 < k^M$) is an unstable steady state.*
2) *Condition $k < k^M$ is equivalent to the following inequalities:*

$$\frac{\left(1-\frac{b}{1+\alpha}\right)y}{k\alpha} > \frac{1}{\mu} \Leftrightarrow \frac{g}{i} > \frac{1-\mu}{\mu}$$

3) *The equilibrium points of dynamic system (9), although unstable, are associated with at least one stable path which converges to this equilibrium (there is a stable manifold at least of dimension 1), as there exists a positive eigenvalue which is lower than $\frac{1}{1+\alpha}$.*

<u>Proof</u>.-

In our Appendix we develop the polynomial characteristic of matrix (21), $Q(\lambda) = \left|J - \lambda I_5\right|$, whose roots are the eigenvalues of this matrix.

Regarding *part 1* of the proposition, and taking into account the results of the Appendix, it is fulfilled that:

$$Q(1) = \frac{\rho\, a_\pi\, \theta\, y}{(1+\alpha)^2(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}\left[\mu\left(1-\frac{b}{1+\alpha}\right)\frac{y}{k}-\alpha\right] > 0$$

$$\Leftrightarrow \mu\left(1-\frac{b}{1+\alpha}\right)\frac{y}{k}-\alpha > 0 \Leftrightarrow \frac{k}{y} < \frac{\mu}{\alpha}\left(1-\frac{b}{1+\alpha}\right)$$

Substituting $y = \left(\frac{\gamma}{\rho}\right)^{1-\mu} k^{\mu}$, we have:

$$Q(1) > 0 \Leftrightarrow k < \left[\frac{\mu}{\alpha}\left(1-\frac{b}{1+\alpha}\right)\right]^{\frac{1}{1-\mu}}\frac{\gamma}{\rho} = k^M$$

Therefore, if $k < k^M$, applying the Bolzano Theorem, we obtain:

$$\left.\begin{array}{l} Q(1) > 0 \\ \\ \lim\limits_{\lambda \to +\infty} Q(\lambda) = -\infty \end{array}\right\} \Rightarrow \exists \lambda \in (1, +\infty) \text{ so that } Q(\lambda) = 0$$

and the equilibrium point $(k, y(k), i(k), \pi(k), R(k))$ defined by $k < k^M$ will be unstable, thus proving *part 1* of the proposition.

Part 2 is easily deduced from the previous equivalences and expressions (12) and (13) as can be seen.

$$k < k^M \Leftrightarrow \frac{k}{y} < \frac{\mu}{\alpha}\left(1 - \frac{b}{1+\alpha}\right) \Leftrightarrow \frac{1}{\frac{\mu}{\alpha}\left(1 - \frac{b}{1+\alpha}\right)} < \frac{y}{k} \Leftrightarrow \frac{g}{i} > \frac{1-\mu}{\mu}.$$

Regarding *the third part*, as we can see in the appendix:

$$\left.\begin{array}{l} Q(0) < 0 \\ \\ Q\left(\frac{1}{1+\alpha}\right) > 0 \end{array}\right\} \Rightarrow \exists \lambda \in (0, \frac{1}{1+\alpha}) \text{ so that } Q(\lambda) = 0$$

Consequently, for any steady state there is always an eigenvalue of matrix (21) with a modulus lower than one, and therefore it has its own associated eigenvector which defines a stable path converging to the equilibrium. Thus *part 3* of the proposition is proven.

It is interesting to ponder the economic meaning of these results. The first section shows us that the steady states such that $k < k^M$, are unstable, although there is a certain degree of stability according to *part three*. Consequently, when $k < k^M$ we must expect two possible evolutions: in general, a deviance from the equilibrium will give way to increasing deviances from this; however, it is also possible that, when we are near the stable path, the deviance takes some time to show itself, giving way to an apparently stable path which eventually will move away from the equilibrium.

Part 2 gives more information about the sources of instability for equilibriums $k < k^M$. To be specific, in our model, the interactions between short, mid and long term mechanisms generate relationships between the steady state investment, the Keynesian multiplier of consumption and technical aspects of production which can generate instability. We can see this by re-writing the first part of the equivalence of *part 2*. For this, taking into account that σ is the family's saving rate, and bear in mind that μ is the production elasticity with regards to *k*:

$$\mu > \frac{\alpha k}{\left(1 - \frac{b}{1+\alpha}\right) y} \Leftrightarrow \mu > \frac{\alpha}{\sigma}\frac{k}{y} \Rightarrow \frac{\partial y}{\partial k} > \frac{1}{\sigma}\frac{\partial i}{\partial k}.$$

This condition is fulfilled if, and only if, we are in the unstable equilibriums associated to $k < k^M$, and has an interesting similarity with certain mechanisms of the Harrod-Domar Keynesian growth model. In this model, the potential production and aggregate demand only grow evenly for a specific rate given for each saving rate, with razor-edge type dynamics appearing in other cases. In our model, the dynamics are much more complex (considering substitutability between the productive factors, labour-increasing technical progress, the Phillips curve compatible with the natural rate, and monetary and fiscal policies). However, the inequalities we have just obtained remind us of the mechanisms in the Harrod model. Thus, we see in the equilibriums associated to $k < k^M$, that the variation of production with respect to k, is higher than the variation generated by the aggregate demand in response to changes in *k*. That is, the effect via capital productivity (faced with an infinitesimal variation of *k*) is *higher* than the multiplier effect produced by said variation of *k* (via investment and effects on consumption) on the aggregate demand.

Can this, in general, entail the instability of this kind of equilibriums? In our model, given that supply adjusts to demand, face with a shock in *k* which can take the system out of an equilibrium $k < k^M$, the variable which adjusts the market is the employment rate ε. In contrast to the Harrod model, in our model there is substitutability between factors (see eq.(5)). For example, face with a shock which would increase *k* (and therefore investment *i*) and take the system out of an equilibrium $k < k^M$, given that the multiplier effect via demand would be insufficient in comparison with the increase generated in the productive capacity, the system would adjust by destroying employment and generating (according to eq.(7)) a fall in the inflation rate. Note that, once out of the equilibrium (although it remains near), the stated effects on income (increase in income) and inflation (fall in inflation) would generate different pressures on the nominal interest rate and the real interest rate (see the Taylor Rule in (8) and its effects on system (9)). In any case, the instability detected by the formal analysis tells us that, for any equilibrium such that $k < k^M$, in general the pressures which tend to amplify the initial shock will prevail, whether this involves an increase or a decrease.

However, looking at *part 3* of the proposition, there is at least one path for which the initial expansive shock in *k* and *i* will turn itself around, via combined effects of monetary policy (minor effects for some paths) and the fall in inflation over the real interest rate (which could end up increasing). In this case, investment could be reduced thus re-establishing the equilibrium position.

Finally, *part 2* of proposition 1 offers us additional information if we observe the second part of the equivalence. Situating ourselves in equilibriums ($k < k^M$) is equivalent to the public spending/private investment ratio being above a certain limit given by the substitutability of the factors – and this means instability. This leads to an interesting political implication: if the authorities maintain the public spending/ private investment ratio above $\frac{1-\mu}{\mu}$, an equilibrium $k > k^M$ will never be reached (Fig.1 right). We shall now study this kind of steady states.

4.2.- Results of stability regarding steady states where $k > k^M$

Remember that the equilibriums associated to $k > k^M$ verify that the savings/investment ratio is lower than limit $\frac{1}{\mu}$, or, what is the same, that the public spending/private investment ratio is lower than $\frac{1-\mu}{\mu}$. That is to say, as we know from the previous sub-section, it can be verified in these equilibriums that:

$$\frac{\sigma}{\alpha}\frac{y}{k} < \frac{1}{\mu} \Rightarrow \frac{1}{\sigma}\frac{\partial i}{\partial k} > \frac{\partial y}{\partial k}.$$

Proposition 2.- *In the case of the equilibrium associated to a value* $k > k^M$ *it is verified that:*

1) *For certain ranges of parametric values the equilibrium is locally asymptotically stable. However, we cannot affirm that there is stability in general, as in many cases some of the eigenvalues of* (21) *have a modulus greater than one.*
2) *Three of the eigenvalues of* (21) *are real; one being negative and two positive, with values lower than one – one of which is lower than* $\frac{1}{1+\alpha}$ *and the other is higher than* $\frac{1}{1+\alpha}$. *Thus, we can state that there are two stable paths (there exists a stable manifold of at least dimension 2). The other two eigenvalues are, either positive real ones, or complex ones with a positive real part.*
3) *If g is sufficiently close to 0, the equilibrium is unstable.*
4) *If we are looking at a stable equilibrium and all the parameters of the model remain constant except sensitivity to inflation in the Taylor Rule, the increase of this sensitivity will eventually make the equilibrium unstable due to overreactions of the monetary policy.*

Proof.-

Part 1 of the above proposition is proven directly by simulations made with Mathematica 10.0, calculating the corresponding eigenvalues. In section 6 of this work we shall present the results of some simulations, confirming that there are equilibriums which are locally asymptotically stable with all the eigenvalues with a modulus below one.

To see the general properties presented in *part 2* of the proposition, we remember that in proposition 1 it was proven that there is an eigenvalue of (21) $\lambda \in \left(0, \frac{1}{1+\alpha}\right)$. It has also been proven that:

$$Q(1) < 0 \Leftrightarrow k > k^M$$

Thus, if $k > k^M$, applying the Bolzano Theorem, we see:

$$\left.\begin{array}{l} Q\left(\frac{1}{1+\alpha}\right) > 0 \\ \\ Q(1) < 0 \end{array}\right\} \Rightarrow \exists \lambda \in \left(\frac{1}{1+\alpha}, 1\right) \text{ so that } Q(\lambda) = 0$$

In addition, given that it is verified that

$$\left.\begin{array}{l} Q(0) = Q_2(0) + Q_5(0) = Q_2(0) < 0 \\ \lim\limits_{\lambda \to -\infty} Q(\lambda) = +\infty \end{array}\right\} \Rightarrow \exists \lambda \in (-\infty, 0)$$

there is also a real negative root, and hence the *first part* of this section is proven.

What is more, as the sum of the five eigenvalues verifies (see Appendix):

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = \frac{1}{1+\alpha} + \frac{b}{1+\alpha} + \alpha\frac{k}{y} + 2 = \frac{1}{1+\alpha} + 3 - \frac{g}{y}$$

if λ_1 is the negative root, $\lambda_2 \in \left(0, \frac{1}{1+\alpha}\right)$ and $\lambda_3 \in \left(\frac{1}{1+\alpha}, 1\right)$, we have:

$$\lambda_4 + \lambda_5 > 2 - \frac{g}{y} + |\lambda_1| > 0$$

And, as the product of these two roots is positive (see Appendix), we can state that both are positive real or complex with real positive parts. This completes *part 2*.

To see *part 3*, it is sufficient to focus on the last inequality obtained. The results found for eigenvalues are also valid for the case $g = 0$, thus verifying:

$$\lambda_4 + \lambda_5 > 2 + |\lambda_1| > 2$$

which is only possible if one of these roots, if they are real, is greater than one; and in the case where they are complex, if their real parts, and therefore their modulus, are greater than one. In addition, as the coefficients of $Q(\lambda)$ are continuous functions of g, the eigenvalues will also be so, which allows us to affirm that if g is positive but small enough, for continuity, it will be verified that:

$$\lambda_4 + \lambda_5 > 2 - \frac{g}{y} + |\lambda_1| > 2$$

Therefore, one of the eigenvalues will have a modulus greater than one and the equilibrium will be unstable as stated in *part 3*.

Regarding *part 4,* multiplying the 5 eigenvalues we obtain the following expression $\frac{-\mu\,\rho\,b\,\theta\,y}{(1+\alpha)^3\,(1-\mu)}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}(1+a_\pi)$, and this expression $\frac{-\mu\,\rho\,b\,\theta\,y}{(1+\alpha)^3\,(1-\mu)}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}$ do not change if all the parameters of the model are maintained except the sensitivity to inflation in Taylor's adjustment. Therefore, it is verified that as a_π grows, there will be a moment at which the product will be, in absolute values, greater than 1. This means that one of the eigenvalues will have a modulus greater than 1 and that, therefore, the equilibrium will be unstable.

Proposition 2 throws up interesting economic intuitions. Firstly, (due to the fact that stable equilibriums only exist when $k > k^M$; that is, when the private savings/investment ratio is below the threshold $\frac{1}{\mu}$, or in other words, that the public spending/private investment ratio is below $\frac{1-\mu}{\mu}$) stability requires a suitable relation between consumption and investment as a necessary condition. Another way of looking at this is that the public spending/private investment ratio must be maintained below a certain level. However, although these conditions are necessary, in general they are not sufficient, as we have seen.

The Harrod-type representation of the equivalences associated to equilibriums $k > k^M$ reveals new implications. Thus, the expression $\frac{1}{\sigma}\frac{\partial i}{\partial k} > \frac{\partial y}{\partial k}$ in $k > k^M$, shows that, faced with a decrease in capital with respect to a stable steady state, *i* will fall, producing a multiplied contractive effect (via consumption) relatively intense; in fact, a fall in income (via demand) larger than the fall in production would be produced, which would force the destruction of employment and so inflation would fall. This decrease in income together with the decrease in inflation could very probably cause a decrease in R (eq.8) which would tend to turn the process back (increase in income and investment, increasing k and re-establishing the initial steady state). However, if the equilibriums $k > k^M$ were big enough for the capital productivity to be almost "0" (a case contemplated in *part 3* of the proposition), the reaction of supply when faced with initial falls in k would be almost null, destroying a lot of employment (and causing the inflation rate to fall significantly). In these cases the fall in demand and the increase in the real interest rate due to the large fall in inflation could cancel out the effects of monetary policy, and so the equilibriums for relatively large k would be unstable.

Proposition 2 also reveals something already seen in proposition 1: the possibility of "apparent" stabilities, which could be misleading. Since there is the possibility of saddle-path type instability, we can find paths which, though they seem to be stable over time (see simulations in section 6), they are actually not stable.

Part 3 of the proposition shows once again the strong relationship between fiscal policy and stability. Levels of "g" which are too low, contribute to the instability of the steady states above k^M. We gave the explanation for this earlier, when speaking about the possible effects of moving far from the equilibrium for sufficiently high k: instability.

As a consequence of the above-mentioned, a certain level of public spending is seen to be necessary in our model. This all leads us to a normative criteria for fiscal policy: the public spending/investment ratio must be below $\frac{1-\mu}{\mu}$ - a necessary condition for stability – but not too much lower, or we face instabilities associated to large k.

Finally, regarding monetary policy, *part 4* of proposition 2 shows another interesting result. Taken with caution, monetary policy acts, in the case of equilibriums above k^M, as an equilibrating factor. However, an excessively forceful use of this could even break up previously-stable situations.

5.- Indecomposability of the model's dynamics

The system of equations (9) describes the general dynamics of the model and suggests a possible decomposability of the system in two autonomous processes: the process of nominal adjustment of the interest rate, and the process of adjustment of the four real variables. These two processes are seen specifically in the Taylor adjustment via *R* (the last equation) and the dynamics of the variables $\{k, y, i, \pi\}$ described by the first four equations.

The adjustment of *R* would move the economy towards values of *y* and π which would verify the equation $a_\pi\left(\pi_t - \pi^*\right) + a_y\left(y_t - y^*\right) = 0$, which captures the trade-off between income and inflation in monetary policy. The other dynamics would basically control the evolution of $\{k, y, i, \pi\}$. If the decomposition were possible, it would be reasonable to expect that if the system were stable, the reduced system of $\{k, y, i, \pi\}$ would also be stable, at least in some cases. We shall see that this, in contrast to that habitually seen in many dynamic models, is not true.

If the Taylor adjustment process had occurred (or worked well in this sense), the value of *R* would be (or would be close to) the constant value of the steady state, and the 4x4 reduced system would have the following characteristic equation:

$$\left|J-\lambda I_4\right| = \begin{vmatrix} \frac{1}{1+\alpha}-\lambda & 0 & \frac{1}{1+\alpha} & 0 \\ 0 & \frac{b}{1+\alpha}+\alpha\frac{k}{y}-\lambda & 0 & \frac{\theta}{1+\alpha}y \\ 0 & \alpha\frac{k}{y} & -\lambda & \frac{\theta}{1+\alpha}y \\ \frac{-\mu\rho}{1-\mu}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}} & \frac{\rho}{1-\mu}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}} & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left(\frac{1}{1+\alpha}-\lambda\right)(-\lambda)\left(\frac{b}{1+\alpha}+\alpha\frac{k}{y}-\lambda\right)$$

$$-\frac{\theta y}{1+\alpha}\frac{1}{1+\alpha}\frac{\mu\rho}{1-\mu}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\left(\frac{b}{1+\alpha}+\alpha\frac{k}{y}-\lambda\right)$$

$$-\frac{\theta y}{1+\alpha}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\frac{k}{y}\frac{\rho}{1-\mu}\left[\lambda^2-\frac{1}{1+\alpha}\lambda-\frac{\alpha\mu}{1+\alpha}\right]$$

We know that $\lim_{\lambda\to+\infty}\left|J-\lambda I_4\right|(\lambda)=+\infty$ and we can also see that the following is verified:

$$\left|J-\lambda I_4\right|(1) = -\frac{\theta y}{1+\alpha}\frac{1}{1+\alpha}\frac{\mu\rho}{1-\mu}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\frac{b}{1+\alpha}+\frac{\theta y}{1+\alpha}\frac{1}{1+\alpha}\frac{\mu\rho}{1-\mu}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}$$

$$+\frac{\theta y}{1+\alpha}\frac{-\alpha}{1+\alpha}\frac{\rho}{1-\mu}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\frac{k}{y} = \frac{\theta y}{1+\alpha}\frac{1}{1+\alpha}\frac{\rho\mu}{1-\mu}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\left[-\frac{b}{1+\alpha}+1-\frac{\alpha}{\mu}\frac{k}{y}\right]<0$$

$$\Leftrightarrow \frac{\mu}{\alpha}\left(1-\frac{b}{1+\alpha}\right)<\frac{k}{y} \Leftrightarrow \frac{\left(1-\frac{b}{1+\alpha}\right)y}{\alpha k}<\frac{1}{\mu} \Leftrightarrow \frac{g}{i}<\frac{1-\mu}{\mu}$$

Therefore, under this condition it would exist a real eigenvalue, positive and greater than one, which would assure that the equilibrium point would be unstable.

It is clear that the previous instability occurs precisely in cases where it is possible to find stable equilibriums, in accordance with Proposition 2; that is, when $k>k^M$.

The conclusion is immediate: it is not possible to separate the processes of convergence towards a stable equilibrium in a process of nominal adjustment of the interest rate, and another of real adjustment of the rest of the variables, thus revealing the strong interdependence of monetary policy and the general growth, employment, etc. dynamics in the model. In other words, we must be cautious when stating with too much conviction that a total independence between Central Banks (drivers of monetary policy) and Governments (who implement stabilization, growth, etc. policies) must exist. Even though it is convenient that there are no subordinations between these institutional spheres, their policies cannot be totally independent.

Finally, note that the previous result also tells us that all kinds of adjustment policy must include margins of flexibility in the nominal interest rate, and it is not possible to have a policy of convergence towards an equilibrium which keeps R constant.

6.- Simulations

The results we obtain clearly show that the equilibriums of the model are not always stable; the equilibrium is unstable when $k<k^M$. Furthermore, as we have seen, even though $k>k^M$, the equilibrium can be unstable if public spending g is sufficiently small. This all leads us to use simulations (as it is impossible to obtain more formal results) to check the existence of *stable steady states* for values of $k \in \left(k^M, k^S\right)$. The simulations were carried out using Mathematica 10.0.

Proven results and implications:

1) *There are stable equilibriums in the model, for values of* $k > k^M$.

   To be specific, for g=0.63; a_π=0.06; a_y=0.015; π^*=0.02; y^*=9.9; β=0.21735; α=0.005885; θ=0.00036; γ=0.00084; μ=0.4; b=0.6965; ρ=0.001; i_0=2; π_0=0.025; R_0=0.023; $k_0=k^S/2$ with y_0 taking its equilibrium value and t_{final}=2000, we have a stable equilibrium, with the highest modulus of the eigenvalue being 0.999539.

2) *The same result is also reached with higher values of a_π and a_y but as they increase, oscillations increase and finally the equilibrium becomes unstable due to over-reaction.*

Multiplying the previous a_π and a_y by factor f_a equal to 1000 leads to oscillations, but the equilibrium remains stable, with the highest modulus of the eigenvalues being 0.999737. If we multiply by f_a=1500, the equilibrium becomes unstable, with the highest modulus of the eigenvalues being 1.00965. The following graphs are for f_a of 1000 and 1500 respectively.

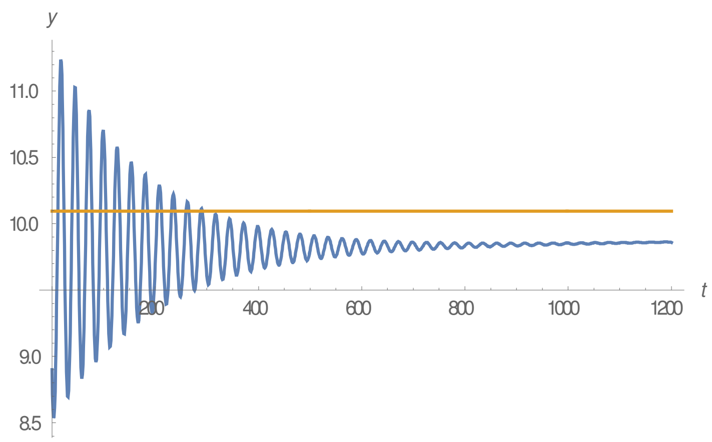

Figure 2.- Stable Steady State but with cyclical oscillations

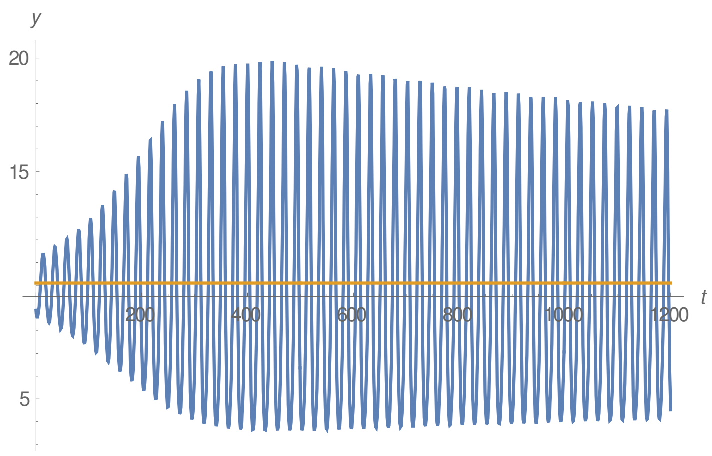

Figure 3.- Unstable Steady State with cyclical oscillations.

3) *The adjustment with a_π is more relevant than that with a_y, but both are strengthened.*

In both previous cases (Fig.2 and Fig.3), with a null value of a_π there is no stable equilibrium, while there is one for a null a_y in the case of Fig.2 (although with oscillations). With a null a_y in the second case (Fig.3) the possible range of f_a is lower; multiplying by 200 instead of 1000 still gives us a stable equilibrium with the highest modulus of the eigenvalues being 0.99949.

4) *The value of g=0.63 is not the only one compatible with stable equilibriums.*

   With the parametric values of point 1, the previous g can be multiplied by, for example, a factor $f_g \in (0.1;1.5)$ without losing stability.

5) *A similar result can be obtained for* μ.

   Setting out from the parametric values of section 1, μ can be multiplied by a factor $f_\mu \in (0.8;1.4)$ without losing stability.

6) *The value of* α *also allows for a range of variation.*

   With the parametric values of section 1, alfa can be multiplied by $f\alpha \in (0.1;2.6)$ without losing stability.

7) *Finally, it is important to point out that simulations can give way to misleading situations. This is thus because an equilibrium can be unstable, but being of a saddle-path type; this can give way to evolutions which seem to be stable but, in fact, are not.*

   This is what happens in the conditions of the first case if we multiply by f_a =1144, which makes the highest modulus of the eigenvalues becoming 1.00005, thus making the equilibrium unstable. However, note the following representations of income y for 30000 iterations (Fig.4) and 200000 (Fig.5).

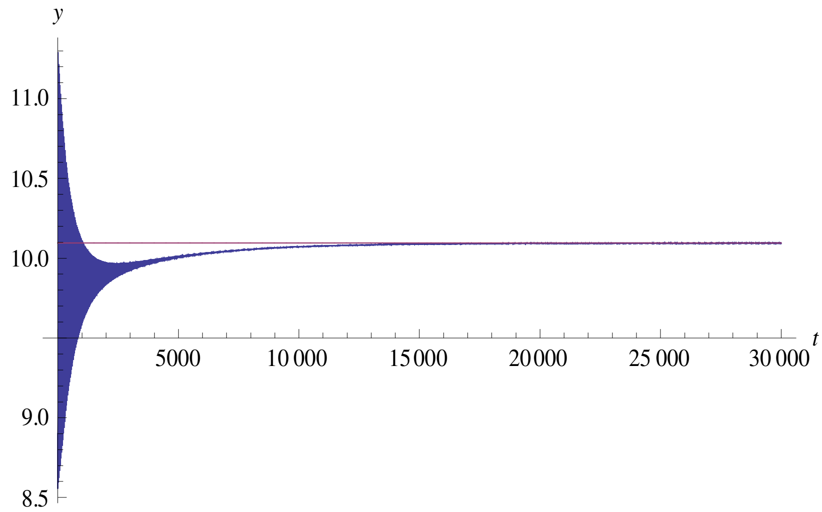

Figure 4.- Apparently stable Steady State

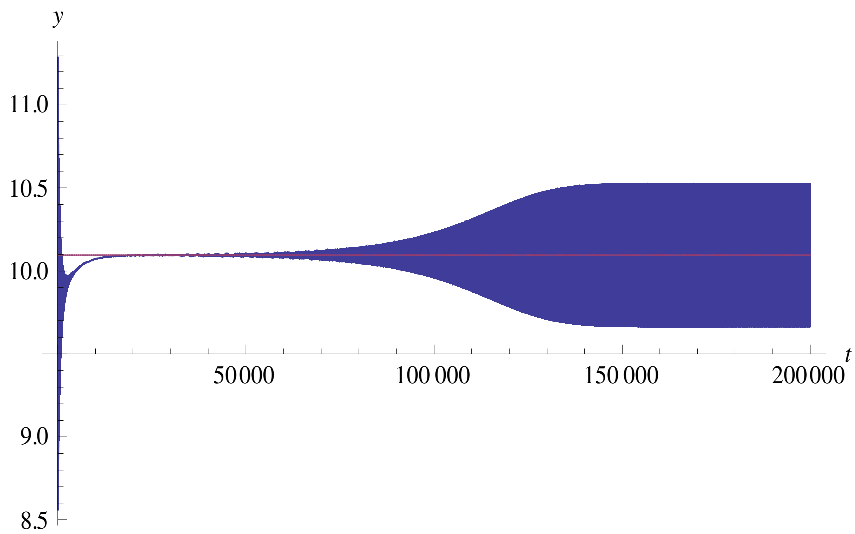

Figure 5.-Increasing volatility around the Steady State for a sufficiently high *t*

The system evolves along an apparently stable path coming close to (without reaching) an unstable steady state. Suddenly, the dynamics start to gain volatility, moving away from the equilibrium. This result could make us rethink episodes such as the Great Moderation and its sudden transformation into the Great Recession.

To sum up, the first conclusion we draw from these simulations is the possible existence of stable equilibriums in the model, as stated in Proposition 2.

It is worth pointing out that these simulations – specifically those associated with result 7 – prove another of the conclusions deduced from Proposition 2; that is, the existence of apparently stable equilibriums which were not actually stable and could mislead econometric contrasts. If we focus on figures 4 and 5, the evolution of income shows a clear stability in the mid term, but not in the long term, which can lead us to interpret long periods of economic history as being stable when in fact they are not.

Let us note, also, that stability could be reached through dampened oscillations as seen in Figure 2. In other words, the cyclical evolution is not anomalous but a usual case – its presence being possible in both stable and unstable situations, as can be seen in Figure 4.

We can also see that the simulations of result 4 confirm the above-stated regarding levels of public spending and their possible influence on economic stability.

Finally, simulations of results 2 and 3 reveal the limits of monetary policy stabilization techniques. Taylor's rule is seen to be operative, and reactions to deviations in inflation are more efficient than reactions to deviations in income. In either case, overacting in this aspect can also be a source of instability according to our simulations.

7.- Conclusions and policy implications

The Great Recession of 2007 obliges us to reflect on its causes and consequences, and leads us to rethink some of the questions regarding macroeconomic policy which had been apparently resolved before the crisis. Our hopes for reaching a stable economy whose instabilities would be easily controlled by economic policy are now much lower than in previous decades. In this work, we proposed a relatively tractable model which, starting out from known hypotheses, allowed us to investigate the origin of some economy instabilities. Based on this model, we have rethought classic questions of macroeconomic policy: the role of monetary policy, the role of fiscal policy, relationships between the short, mid, and long term, and the conditions of stability and instability on which advanced economies rest. The relative simplicity of the model has allowed us to obtain results of existence and local stability of steady states, and to design interpretable simulations.

Regarding our results, we can confirm from our model that certain pre-crisis prescriptions of Macroeconomic policy were less "unquestionable" than previously thought. The formal analysis of the model reveals that there are multiple equilibriums with different characteristics of stability and instability. Far from the usual conviction in mainstream macroeconomics that steady states are “acceptably stable”, our analysis shows many sources of instability. In this sense, it is interesting to note that, although under certain conditions monetary policy helps to stabilize the dynamics in our model, in other circumstances we show it can be a source of instability. Moreover, in our model, it is possible that - dependent on the objectives of the Central Bank - the economy can become stuck in situations similar to the liquidity trap, and so the need to apply non-conventional monetary policies appears.

Another interesting result is that fiscal policy plays a very relevant role in our model, not only in the short term but also when attempting to bring the economy nearer any of the multiple (alternative) steady states. Therefore, it is an instrument which can turn out to be essential in the long term (contrary to predictions of many mainstream models). Furthermore, fiscal policy interacts in our model with factors normally considered to be long term factors (technical progress, capital use intensity, population growth) by determining the parametric ranges of stability/instability for the equilibriums and the specific pattern for dynamic behavior (dampened oscillations, or not, saddle-path type instability, etc.).

Finally, the formal analysis of the stability of the model, and the simulations, show that there exist ranges of the parametric space in which the equilibriums are unstable; as well as other ranges in which the system has trajectories which converge to asymptotically stable equilibriums; and others where the system seems to stabilize, suddenly entering a regime of instability and increasing volatility. This is down to the existence of stable paths in a context of saddle-path type instability. These characteristics of the dynamics lead us to, at least, consider the possibility that the Great Moderation might not have been such a stable period as was previously thought, but more a process of approximation to a steady state which, perhaps having saddle-path type stability characteristics, ended up endogenously developing an increase in volatility and instability which then gave way to the Great Recession.

Appendix.- We calculate the characteristic polynomial of (21), developing $\left|J-\lambda I_5\right|$ for the third column:

$$Q(\lambda)=\left|J-\lambda I_5\right|=\frac{1}{1+\alpha}\begin{vmatrix} 0 & \dfrac{b}{1+\alpha}+\alpha\dfrac{k}{y}-\lambda & \dfrac{\theta}{1+\alpha}y & -\dfrac{\theta}{1+\alpha}y \\ 0 & \alpha\dfrac{k}{y} & \dfrac{\theta}{1+\alpha}y & -\dfrac{\theta}{1+\alpha}y \\ \dfrac{-\mu\rho}{1-\mu}\left(\dfrac{y}{k}\right)^{\frac{1}{1-\mu}} & \dfrac{\rho}{1-\mu}\left(\dfrac{y}{k}\right)^{\frac{\mu}{1-\mu}} & 1-\lambda & 0 \\ 0 & a_y & a_\pi & 1-\lambda \end{vmatrix}$$

$$-\lambda\begin{vmatrix} \dfrac{1}{1+\alpha}-\lambda & 0 & 0 & 0 \\ 0 & \dfrac{b}{1+\alpha}+\alpha\dfrac{k}{y}-\lambda & \dfrac{\theta}{1+\alpha}y & -\dfrac{\theta}{1+\alpha}y \\ \dfrac{-\mu\rho}{1-\mu}\left(\dfrac{y}{k}\right)^{\frac{1}{1-\mu}} & \dfrac{\rho}{1-\mu}\left(\dfrac{y}{k}\right)^{\frac{\mu}{1-\mu}} & 1-\lambda & 0 \\ 0 & a_y & a_\pi & 1-\lambda \end{vmatrix}$$

$$\overset{(1)}{=}\frac{1}{1+\alpha}\frac{-\mu\rho}{1-\mu}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\begin{vmatrix} \dfrac{b}{1+\alpha}+\alpha\dfrac{k}{y}-\lambda & \dfrac{\theta}{1+\alpha}y & -\dfrac{\theta}{1+\alpha}y \\ \alpha\dfrac{k}{y} & \dfrac{\theta}{1+\alpha}y & -\dfrac{\theta}{1+\alpha}y \\ a_y & a_\pi & 1-\lambda \end{vmatrix}$$

$$-\lambda\left(\frac{1}{1+\alpha}-\lambda\right)\begin{vmatrix} \dfrac{b}{1+\alpha}+\alpha\dfrac{k}{y}-\lambda & \dfrac{\theta}{1+\alpha}y & -\dfrac{\theta}{1+\alpha}y \\ \dfrac{\rho}{1-\mu}\left(\dfrac{y}{k}\right)^{\frac{\mu}{1-\mu}} & 1-\lambda & 0 \\ a_y & a_\pi & 1-\lambda \end{vmatrix}$$

$$\overset{(2)}{=}\frac{-\mu\rho}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\begin{vmatrix} \dfrac{b}{1+\alpha}-\lambda & 0 & 0 \\ \alpha\dfrac{k}{y} & \dfrac{\theta}{1+\alpha}y & -\dfrac{\theta}{1+\alpha}y \\ a_y & a_\pi & 1-\lambda \end{vmatrix}$$

$$-\lambda\left(\frac{1}{1+\alpha}-\lambda\right)\left[\begin{aligned}&\left(\frac{b}{1+\alpha}+\alpha\frac{k}{y}-\lambda\right)(1-\lambda)^2-\frac{\theta\rho a_\pi y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}\\&+\frac{\theta a_y y}{1+\alpha}(1-\lambda)-\frac{\theta\rho y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}(1-\lambda)\end{aligned}\right]$$

$$= \frac{-\mu\rho}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\left(\frac{b}{1+\alpha}-\lambda\right)\left[\frac{\theta}{1+\alpha}y(1-\lambda)+\frac{\theta a_\pi}{1+\alpha}y\right]$$

$$-\lambda\left(\frac{1}{1+\alpha}-\lambda\right)\left[\begin{array}{l} -\lambda^3+\left(\frac{b}{1+\alpha}+\alpha\frac{k}{y}+2\right)\lambda^2-\left[1+\frac{2b}{1+\alpha}+2\alpha\frac{k}{y}+\frac{\theta a_y y}{1+\alpha}-\frac{\theta\rho y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}\right]\lambda \\ +\frac{b}{1+\alpha}+\alpha\frac{k}{y}-\frac{\theta\rho a_\pi y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}+\frac{\theta a_y y}{1+\alpha}-\frac{\theta\rho y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}} \end{array}\right]$$

(1) The first determinant is developed in the first column, and the second in the first row.

(2) In the first determinant we subtract the second row from the first row; and the second determinant is calculated using the Sarrus rule.

The polynomial can be decomposed as a sum of two polynomials, one of degree 2 and one of degree 5, $Q(\lambda)=Q_2(\lambda)+Q_5(\lambda)$ where:

$$Q_2(\lambda)=\frac{-\mu\rho}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\left(\frac{b}{1+\alpha}-\lambda\right)\left[\frac{\theta}{1+\alpha}y(1-\lambda)+\frac{\theta a_\pi}{1+\alpha}y\right]$$

$$Q_5(\lambda)=-\lambda\left(\frac{1}{1+\alpha}-\lambda\right)\left[\begin{array}{l} -\lambda^3+\left(\frac{b}{1+\alpha}+\alpha\frac{k}{y}+2\right)\lambda^2-\left[1+\frac{2b}{1+\alpha}+2\alpha\frac{k}{y}+\frac{\theta a_y y}{1+\alpha}-\frac{\theta\rho y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}\right]\lambda \\ +\frac{b}{1+\alpha}+\alpha\frac{k}{y}-\frac{\theta\rho a_\pi y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}+\frac{\theta a_y y}{1+\alpha}-\frac{\theta\rho y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}} \end{array}\right]$$

It is verified that $Q_2(\lambda)$ is a concave parabolic whose roots are $t_1=\frac{b}{1+\alpha}$ y $t_2=1+a_\pi$,

while $Q_5(\lambda)$ has, among its roots, $z_1=0$ y $z_2=\frac{1}{1+\alpha}$.

Consequently, applying the Bolzano theorem in the following situations, we deduce that:

$$\left.\begin{array}{l}\lim_{\lambda\to-\infty} Q(\lambda)=+\infty \\ Q(0)=Q_2(0)+Q_5(0)=Q_2(0)<0\end{array}\right\}\Rightarrow \exists\lambda_1\in(-\infty,0)\ \text{ tal que } Q(\lambda_1)=0$$

$$\left.\begin{aligned}&Q(0)=Q_2(0)+Q_5(0)=Q_2(0)<0\\&Q\left(\frac{1}{1+\alpha}\right)=Q_2\left(\frac{1}{1+\alpha}\right)+Q_5\left(\frac{1}{1+\alpha}\right)=Q_2\left(\frac{1}{1+\alpha}\right)>0\end{aligned}\right\}\Rightarrow\exists\lambda_2\in\left(0,\frac{1}{1+\alpha}\right)\text{ tal que } Q(\lambda_2)=0$$

$$\left.\begin{aligned}&Q\left(\frac{1}{1+\alpha}\right)=Q_2\left(\frac{1}{1+\alpha}\right)+Q_5\left(\frac{1}{1+\alpha}\right)=Q_2\left(\frac{1}{1+\alpha}\right)>0\\&\lim_{\lambda\to+\infty}Q(\lambda)=-\infty\end{aligned}\right\}\Rightarrow\exists\lambda_3\in\left(\frac{1}{1+\alpha},+\infty\right)\text{ tal que } Q(\lambda_3)=0$$

We can say the following regarding the other two eigenvalues:

Grouping together the same degree terms in $Q(\lambda)=\left|J-\lambda I_5\right|$ and multiplying by -1, we obtain the characteristic equation of (21), written as

$P(\lambda)=\lambda^5+a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5=0$, where:

$$a_1=-\left(\frac{1}{1+\alpha}+\frac{b}{1+\alpha}+\alpha\frac{k}{y}+2\right)$$

$$a_2=\frac{b}{(1+\alpha)^2}+\frac{\alpha}{1+\alpha}\frac{k}{y}+\frac{2}{1+\alpha}+1+\frac{2b}{1+\alpha}+2\alpha\frac{k}{y}+\frac{\theta a_y\, y}{1+\alpha}-\frac{\theta\rho y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}$$

$$a_3=-\left(\begin{aligned}&\frac{b}{1+\alpha}+\alpha\frac{k}{y}-\frac{\theta\rho a_\pi y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}+\frac{\theta a_y\, y}{1+\alpha}-\frac{\theta\rho y}{(1+\alpha)(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}+\frac{1}{1+\alpha}+\\&\frac{2b}{(1+\alpha)^2}+\frac{2\alpha}{1+\alpha}\frac{k}{y}+\frac{\theta a_y\, y}{(1+\alpha)^2}-\frac{\theta\rho y}{(1+\alpha)^2(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}-\frac{\mu\theta\rho y}{(1+\alpha)^2(1-\mu)}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\end{aligned}\right)$$

$$a_4=\frac{b}{(1+\alpha)^2}+\frac{\alpha}{1+\alpha}\frac{k}{y}+\frac{\theta a_y\, y}{(1+\alpha)^2}-\frac{\theta\rho a_\pi y}{(1+\alpha)^2(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}+\frac{\theta a_y\, y}{(1+\alpha)^2}-\frac{\theta\rho y}{(1+\alpha)^2(1-\mu)}\left(\frac{y}{k}\right)^{\frac{\mu}{1-\mu}}-$$
$$\frac{\mu\rho\theta y}{(1+\alpha)^2(1-\mu)}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}\left(1+a_\pi+\frac{b}{1+\alpha}\right)$$

$$a_5=\frac{\mu\rho b\theta y}{(1+\alpha)^3(1-\mu)}\left(\frac{y}{k}\right)^{\frac{1}{1-\mu}}(1+a_\pi)$$

It is verified that $a_5=(-1)^5\lambda_1\cdot\lambda_2\cdot\lambda_3\cdot\lambda_4\cdot\lambda_5$, thus, taking into account that $a_5>0$, and the signs of λ_1,λ_2 y λ_3, we deduce that $\lambda_4\cdot\lambda_5>0$ and, consequently, either the eigenvalues λ_4 y λ_5 are real with the same sign or are complex. It is also verified that $\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5=-a_1$, thus $\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5=\frac{1}{1+\alpha}+\frac{b}{1+\alpha}+\alpha\frac{k}{y}+2.$

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